

## Chapter 22 Appendix

### A Review of Some Statistics about Portfolio Returns

At its root, portfolio theory is about the risk and expected return in a combination of individual assets. Risk and return are quantified using statistics that describe periodic returns series over time. Therefore, some basic statistics about periodic returns of portfolios are reviewed in this appendix. As portfolios are combinations of individual assets, it is also important to understand the relationship between time-series and cross-sectional return statistics as applied to real estate assets.

#### Cross-Sectional and Time-Series Return Statistics

Return statistics measure the central tendencies and dispersion in the returns to assets or portfolios (combinations of assets). The two types are defined here:

- *Cross-sectional* statistics are taken *across multiple individual assets, as of a single point in time*.
- *Time-series* statistics are taken *across multiple periods of time* (either for a single asset or portfolio, or for multiple assets or portfolios, but always measuring effects *across time*).

Although classical, formal portfolio theory makes direct use only of time-series statistics, both types of statistics are broadly relevant in the analysis and characterization of real estate portfolios. Furthermore, a good understanding of the distinction between these two types of return statistics is necessary to avoid confusion.

The best way to cement your understanding of the difference between these two types of statistics in practice is to consider a simple but concrete numerical example. Accordingly, let us consider two individual real estate assets (two properties), A and B. We will consider their investment performance as represented by their total periodic returns in each of two consecutive periods of time, year 1 and year 2. (To simplify the numerical example, we will assume that neither property generates any income in either year, so that their total returns are determined

entirely by their change in value, or appreciation return.) The following table presents the value of each property at the beginning and end of each year:

### Example Property Values

Time	Value Property A	Property B
Beg. Yr. 1	\$2,000,000	\$1,000,000
End Yr. 1	\$2,160,000	\$1,060,000
End Yr. 2	\$2,419,200	\$1,123,600

Based on these values, we can compute property A's periodic returns as 8 percent and 12 percent, respectively, for year 1 and year 2.<sup>1</sup> Similarly, property B's return is 6 percent in both years.<sup>2</sup> Looking *across time* at each property in turn, we can see that property A's average return was 10 percent and property B's was 6 percent during the period of history encompassed by years 1 and 2 together.<sup>3</sup> These time-series average statistics present a measure of the "central tendency" of the returns to each of these properties during that period. On the other hand, looking *across properties* within each year, we can see that the average return in year 1 was 7 percent and the average return in year 2 was 9 percent.<sup>4</sup> (Note that these cross-sectional average

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<sup>1</sup>  $(\$2,160,000 - \$2,000,000)/\$2,000,000 = 8\%$ .  $(\$2,419,200 - \$2,160,000)/\$2,160,000 = 12\%$ .

<sup>2</sup>  $(\$1,060,000 - \$1,000,000)/\$1,000,000 = 6\%$ .  $(\$1,123,600 - \$1,060,000)/\$1,060,000 = 6\%$ .

<sup>3</sup> For property A:  $(8\% + 12\%)/2 = 10\%$ . For property B:  $(6\% + 6\%)/2 = 6\%$ . Recall from Chapter 3 that these are "arithmetic" means. Arithmetic means tend to be more widely used in portfolio theory. Property A's "geometric" mean return across the two years would be  $(\$2,419,200/\$2,000,000)^{(1/2)} - 1 = 9.98\%$ . Property B's geometric mean return is identical to its arithmetic mean, as it has no dispersion in its return over time:  $(\$1,123,600/\$1,000,000)^{(1/2)} - 1 = 6\%$ .

<sup>4</sup> For year 1:  $(8\% + 6\%)/2 = 7\%$ . For year 2:  $(12\% + 6\%)/2 = 9\%$ .

returns are *equally weighted* mean returns. The weight of each property's return in computing the cross-sectional average is identical, namely, 50 percent.)

We can also compute statistics of dispersion in the periodic returns, either across time or across properties. The simplest statistic of dispersion is the population standard deviation. This is computed as the square root of the equally weighted mean of the squared deviation of each return from its equally weighted mean. For example, by this measure the time-series dispersion in property A's return is  $\pm 2\%$  around its mean of 10 percent, computed as

$$2\% = \sqrt{\left\{ \left[ (8\% - 10\%)^2 + (12\% - 10\%)^2 \right] / 2 \right\}}$$

The cross-sectional dispersion in year 1 would similarly be measured as  $\pm 1\%$  around its mean of 7 percent, computed as

$$1\% = \sqrt{\left\{ \left[ (8\% - 7\%)^2 + (6\% - 7\%)^2 \right] / 2 \right\}}$$

The following table summarizes the individual periodic returns and the previously described mean and standard deviation statistics for our example. In the top left panel, the rows correspond to the two years, and the columns to the two properties. The crosssectional statistics are presented to the right, and the time-series statistics at the bottom. (Note that property B's returns had zero percent standard deviation, indicating no dispersion during these two years, as the return was constant at 6 percent both years.)

	Periodic Returns		Cross-Sectional Statistics	
Year	Property A	Property B	Mean	St. Dev. (Pop.)
1	8.00%	6.00%	7.00%	1.00%
2	12.00%	6.00%	9.00%	3.00%

Time-Series Statistics	
Property A	Property B
Mean 10.00%	6.00%
Std. Dev. 2.00%	0.00%

It is obvious from this simple numerical example that cross-sectional statistics and time-series statistics are very different animals. They are measuring conceptually different things. For example, you should see that it would be conceptually possible for there to be great cross-sectional dispersion within each period of time, yet the returns across time within all (or a combination of) properties could be constant, lacking any time-series dispersion (i.e., no volatility). Such a result would follow if each property always displayed a constant return, but that return was different for every property. Similarly, it would be possible for there to be no cross-sectional dispersion at all (all properties always having the same return within any given period of time), yet a portfolio consisting of all (or any combination of) such properties could have plenty of dispersion in its periodic returns over time (i.e., lots of volatility). All it would take to accomplish this result would be for each property's returns to vary across time.

Although portfolio theory uses time-series statistics, it is most concerned about such statistics for portfolios consisting of multiple individual assets. If each asset in the portfolio always had the same proportion of the portfolio's total value as every other asset, then the cross-sectional equally weighted mean returns such as those computed earlier would represent the portfolio's periodic returns. In practice, however, the value weights of the individual assets in a portfolio usually vary across the assets. For example, consider the portfolio consisting of property A and property B. The values of this portfolio at the beginning and end of each year are shown in the following table, along with the portfolio's periodic returns. The previously described time-series statistics are also computed for the portfolio, based on its periodic returns. The portfolio's time-weighted arithmetic mean return is 8.68 percent across the two years, and its volatility during that period was 1.35 percent.

<b>Time</b>	<b>Property A</b>	<b>Property B</b>	<b>Portfolio</b>	<b>Return</b>
Beg. Yr. 1	\$2,000,000	\$1,000,000	\$3,000,000	
End Yr. 1	\$2,160,000	\$1,060,000	\$3,220,000	7.33%
End Yr. 2	\$2,419,200	\$1,123,600	\$3,542,800	10.02%
T.S. Mean				8.68%
T.S. St. Dev. (Pop.)				1.35%

Note that the portfolio returns of 7.33 percent and 10.02 percent in years 1 and 2, respectively,

differ from the equally weighted average returns computed previously (7 percent and 9 percent). This is because the assets composing the portfolio had different returns and different value-based weights in the portfolio. The portfolio returns may also be referred to as the *value-weighted* average returns across the individual assets. For example, based on the asset values as of the beginning of year 1, property A has two-thirds of the portfolio value and property B has one-third because property A was worth \$2 million at the beginning of year 1, when property B was only worth \$1 million. Thus, the value-weighted average return across the two properties in year 1 is  $(2/3)8\% + (1/3)6\% = 7.33\%$ .

In general, the return during period  $t$  to a portfolio consisting of  $N$  component assets is the value-weighted average return across all the assets:

$$r_{Pt} = \sum_{i=1}^N w_{it} r_{it}, \quad (\text{A.1})$$

where

$$\sum_{i=1}^N w_{it} = 1$$

where  $w_{it}$  is the proportion of portfolio value in asset  $i$  at the beginning of period  $t$ , and  $r_{it}$  is asset  $i$ 's return that period.<sup>5</sup>

### Time-Series Statistics Used in Portfolio Theory

As noted at the outset of this appendix, portfolio theory is essentially about the risk and return characteristics of portfolios. Such risk and return is quantified by statistics describing the period-by-period total returns to the portfolio, returns like those we just calculated for the portfolio in the simplified numerical example in the preceding section.<sup>6</sup> The statistics used directly in portfolio theory are therefore time-series statistics, that is, measures of central tendencies, dispersion, and comovement across time. In practice, any given historical period is normally

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<sup>5</sup> This is equivalent to the weighted average cost of capital (WACC) formula introduced in Chapter 12. As noted there, it applies to the simple holding period return (HPR).

<sup>6</sup> Recall from Chapter 3 that periodic returns are series of consecutive short holding period returns (HPRs) across time.

taken to represent a sample of time. Thus, in practice, we usually use sample statistics rather than the population statistics used in the previous numerical example.<sup>7</sup>

Broadly speaking, we can group the statistics used in basic portfolio theory into two types. *First moment* statistics measure *central tendencies*, that is, the overall average or typical value of the return. *Second moment* statistics measure *dispersion* and *comovement*, that is, how returns vary over time and in relation to those of other assets. The classical first moment statistic used in portfolio theory is the mean, usually the arithmetic mean.<sup>8</sup> The major second moment statistics used in portfolio theory are the variance (and its square root, the standard deviation, or volatility) and the covariance (and its normalization, the correlation coefficient). These statistics are described here.

**Mean** This is the average value of the returns across time. It represents expected performance (ex ante) or achieved performance (ex post). In general, for portfolio theory, the time-weighted mean is most relevant, for the reasons described in Chapter 3. Assuming the historical period from which the mean is calculated is a representative stretch of history, the historical arithmetic mean is the best estimate of the ex ante return, that is, what the return will be in any given future single period.<sup>9</sup> As noted in Chapter 3, the formula for computing the arithmetic mean of a series of  $T$  periodic returns ( $r_1, r_2, \dots, r_T$ ) is as follows, where  $\bar{r}$  is the ex post mean return:

$$\bar{r} = \left( \frac{1}{T} \right) \sum_{t=1}^T r_t$$

The ex ante return, or expected return, labeled  $E[r]$ , may or may not be taken to equal the ex post mean return from a given historical sample.

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<sup>7</sup> The distinction between sample and population statistics is normally only a technical fine point in most of the practical applications we will consider. For example, as noted in formula (A.3), the formula for computing historical return variance (or volatility) is modified slightly from what was used in the previous numerical example.

<sup>8</sup> See Chapter 3 for a discussion of the differences between, and the relative advantages and disadvantages of, the arithmetic and geometric means. Recall also that when returns are defined as continuously compounded (or log differences), then the arithmetic mean is effectively the same as the geometric mean.

<sup>9</sup> The time-weighted geometric mean may be a better indicator of multiperiod average performance over an interval of time equal to that in the historical sample. (Recall that if the underlying returns are normally distributed, the arithmetic mean approximately equals the geometric mean plus one-half the variance in the returns.)

**Variance** This is the most widely used measure of the magnitude of dispersion about the mean.<sup>10</sup> In theory, it is the central tendency of the squared difference between the realized return and the ex ante mean return,  $E[(r - E[r])^2]$ . In practice, variance is often estimated statistically from a historical sample of periodic returns, using the following formula, where  $S^2$  represents the variance:<sup>11</sup>

$$S^2 = \frac{\sum_{t=1}^T (r_t - \bar{r})^2}{T-1} \quad (\text{A.3})$$

**Standard Deviation** This is the square root of the variance. Applied to a series of periodic asset returns as in portfolio theory, this measure is commonly referred to as the volatility. The advantage of this measure over the variance is that it is measured in units of returns rather than “squared returns.” For example, if a return has an expected value of 10 percent based on a 50 percent chance that it will turn out to be 15 percent and an equal chance that it will turn out to be 5 percent, its standard deviation or volatility is 5 percent, while its variance is 25 “percent-squared.” As you can see, the standard deviation is a more intuitive measure than the variance.

As noted in previous chapters, the volatility is a basic measure of the risk of an investment. In particular, the volatility is often taken as the measure of the *total risk* in an investment (that is, including both the “idiosyncratic” or specific risk that can be diversified away and the “systematic” risk that cannot be diversified away). The volatility of an asset or portfolio is often

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<sup>10</sup> Other measures that are sometimes used in portfolio theory instead of the variance include most prominently the mean absolute difference (MAD) and the semivariance (based only on the left-hand tail of the return distribution, that is, only deviations *below* the mean are included). The former is less sensitive to a few “outlier” events, while the latter is arguably a better measure of how most investors view risk if returns are not symmetrically distributed. In practice, however, the traditional variance is still the most widely used measure of risk, and for many portfolio applications will give results similar to those obtained from other measures.

<sup>11</sup> Note that the sum of the squared differences from the mean are divided by *one less than* the number of observations in the time-series. This modification is necessary to remove bias from statistical estimators of second moments. In effect, this formulation of the variance estimator is treating the historical time-series as a sample drawn from a larger potential population of returns (including those that will occur in the future). In implementing this formula in an electronic spreadsheet (such as Excel), you need to use the “sample” (as opposed to “population”) version of the formula:  $\text{VAR}()$  as opposed to  $\text{VARP}()$ .

used to measure the amount of risk an undiversified investor (that is, one who holds only that asset or portfolio) is exposed to.

The formula for estimating volatility from a historical series of periodic returns, labeled  $S$ , is as follows:

$$S = \frac{\sum_{t=1}^T (r_t - \bar{r})^2}{T-1} \quad (\text{A.4})$$

**Covariance** This is the most widely used measure of pairwise comovement, that is, how the returns of two assets (or portfolios) move together across time. What the variance measures for a single asset's returns, covariance measures for a pair of assets.<sup>12</sup> The formula for computing the historical covariance between asset (or portfolio)  $i$  whose returns are  $(r_{i1}, r_{i2}, \dots, r_{iT})$  and asset (or portfolio)  $j$  whose returns are  $(r_{j1}, r_{j2}, \dots, r_{jT})$ , labeled  $COV_{ij}$ , is as follows:<sup>13</sup>

$$COV_{ij} = \frac{\sum_{t=1}^T [(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)]}{T-1} \quad (\text{A.5})$$

In terms of its economic significance in portfolio theory, the most important thing about the covariance statistic is the fact that the covariance between an asset and a portfolio is the component of that asset's variance that is not diversified away when the asset is added to the portfolio. Thus, covariance is basic to measuring the “systematic risk” in an asset, the risk that cannot be diversified away. This characteristic of covariance may help you to see the intuition in the following formula, which gives the time-series variance of a portfolio ( $S_p^2$ , the “volatility-squared” of the portfolio) as a function of the covariances of the returns to the  $N$  constituent assets in the portfolio and the weight ( $w_i$ , the share of the total portfolio value) invested in each

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<sup>12</sup> Indeed, covariance is a generalization of the variance measure, as one way to define the variance is as the covariance of an asset with itself.

<sup>13</sup> By comparing formula (A.5) with formula (A.3), you can see that  $COV_{ii}$ , the covariance of asset  $i$  with itself, is equivalent to the variance of asset  $i$ :  $S_i^2$ .



asset  $i$  ( $i = 1, \dots, N$ ):<sup>14</sup>

$$S_p^2 = \sum_{i=1}^N \sum_{j=1}^N w_i w_j COV_{ij} \quad (\text{A.6})$$

In other words, the variance of a portfolio is the sum of the products of the weighted covariances of the individual constituent assets across all of the pairs of assets in the portfolio.<sup>15</sup>

This is the formula that was used in the table in Exhibit 21-1C to compute the volatility of Bob's portfolio.

**Cross-Correlation Coefficient** This statistic (referred to simply as the correlation for short) is a normalized measure of comovement. While the covariance is measured in units of “squared return” like the variance, the correlation ranges between negative unity and positive unity. The correlation (labeled  $C_{ij}$ ) equals the covariance divided by the product of the standard deviations of the two variables:

$$C_{ij} = \frac{COV_{ij}}{S_i S_j} \quad (\text{A.7})$$

Note that the correlation coefficient has the same sign as the covariance, but it is bounded between  $-1$  and  $+1$ .

Exhibit 22A-1 pictures what different correlations would typically look like in scatter- plots of the two assets' returns. Exhibit 22A-1A depicts  $+90\%$  correlation. Exhibit 22A-1B depicts  $-$

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<sup>14</sup> Recall from formula (A.1) that the weights sum to one:

$$\sum_{i=1}^N w_i = 1$$

<sup>15</sup> This is a summation of  $N^2$  terms. Note, however, that  $w_i w_j COV_{ij} = w_j w_i COV_{ji}$ , and recall that  $COV_{ij} = VAR_i = S_i^2$ , the variance of asset  $i$ . Thus, formula (A.6) can also be expressed as a summation of  $N + [(N^2 - N)/2]$  terms, including  $N$  terms in the variances of the assets and  $(N^2 - N)/2$  terms that are twice the covariances of each different pair of assets in the portfolio:

$$S_p^2 = \sum_{i=1}^N w_i^2 S_i^2 + 2 \sum_{i=1}^N \sum_{j=1+1}^N w_i w_j COV_{ij}$$

90% correlation. With *positive* correlation, the two assets' returns tend to move *together*: when one is up, the other tends also to be up; when one is down, the other tends also to be down. *Negative* correlation indicates that the two assets' returns tend to move *oppositely*: when one is up, the other is down. Zero correlation indicates a lack of systematic relationship between the two assets' returns. As the correlation approaches positive or negative 100 percent, the relationship between the two assets approaches a *deterministic* relationship in which the two assets move in lockstep (always either together or oppositely, lying on a perfectly straight line of either positive or negative slope<sup>16</sup>).

The correlation is important in portfolio analysis because it indicates how useful an asset (or a pair of assets) can be for purposes of diversification. The lower the correlation coefficient between two assets is, the better they can “diversify each other,” in the sense that the greater will be the potential to reduce volatility by combining the two assets in a portfolio. In the extreme, two assets that are perfectly negatively correlated ( $C_{ij} = -100\%$ ) can be combined to produce a portfolio that is completely riskless, in the sense of having zero volatility.

**Autocorrelation** This is simply the correlation of an asset's (or portfolio's, or asset class's) returns with themselves lagged in time. The autocorrelation reflects the nature of the informational efficiency of an asset market, as discussed in previous chapters. For example, in general, zero autocorrelation is an indication of an informationally efficient asset market, in which prices quickly reflect full information (and therefore, returns lack predictability). Positive autocorrelation indicates a sluggish asset market that lacks perfect informational efficiency (prices only gradually incorporate new information). Negative autocorrelation indicates a “noisy” market (excessive short-run volatility, that is, a tendency for price overreactions that are subsequently “corrected”).

Evidence suggests that private property markets display positive autocorrelation in the short to intermediate term. Public securities markets (including REITs) tend to show very little

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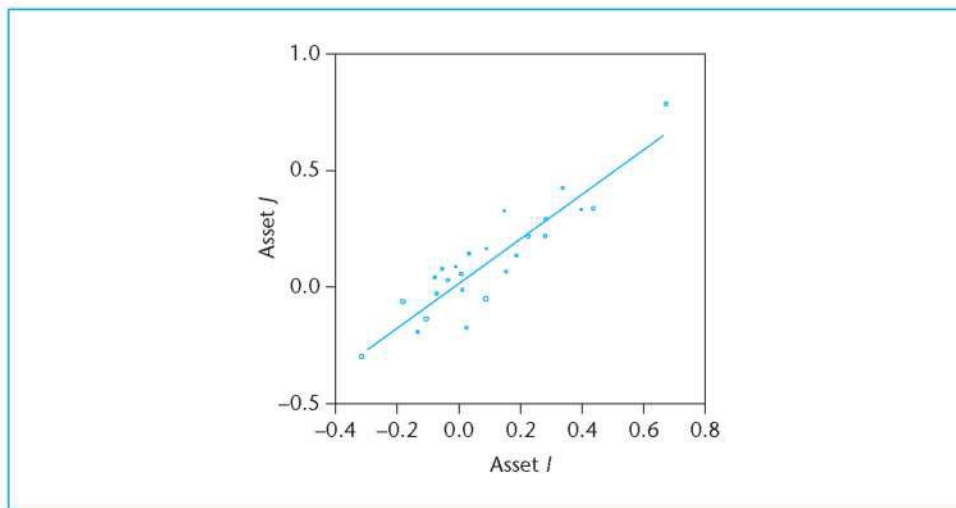
<sup>16</sup> The correlation indicates the sign of the slope but not its magnitude or steepness.

autocorrelation in general, though with some tendency toward negative autocorrelation in the intermediate run (perhaps offsetting some positive autocorrelation in the very short run).

When an asset has zero autocorrelation (over all lags), the variance in its return is directly proportional to the length of the time interval over which the return is measured. For example, annual periodic returns would have four times the variance of quarterly periodic returns. When an asset has positive autocorrelation, the long-run variance is more than proportional to the short-run variance, while the opposite is true with negative autocorrelation. Thus, private real estate volatility tends to increase *relative to* that of securities when returns are measured at lower frequency (longer periodic intervals). For example, quarterly private real estate returns generally appear quite a bit less volatile than quarterly stock market returns, but five-year-interval private real estate returns are probably about as volatile as five- year-interval stock market returns.

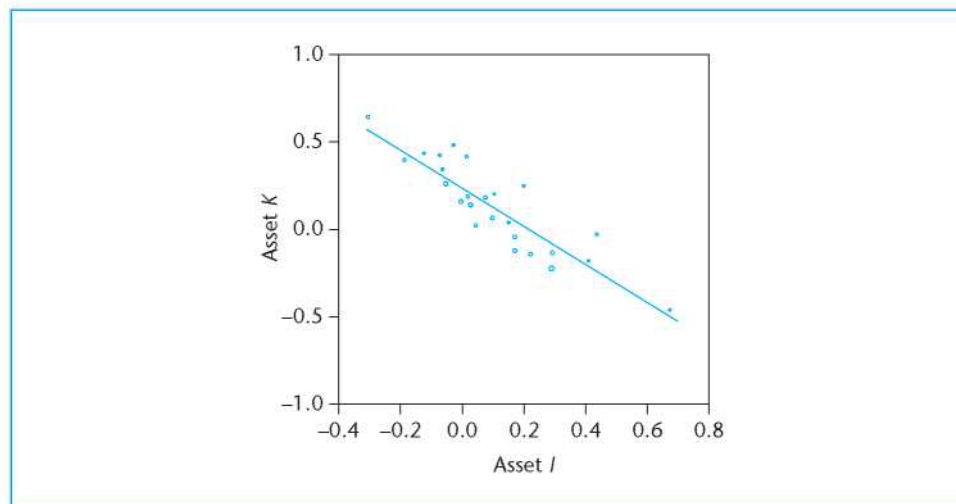
### EXHIBIT 22A-1A

+90% Correlation



### EXHIBIT 22A-1B

-90% Correlation



## EXHIBIT 22A-1C

0% Correlation

